Algebraically explicit analytical solutions of unsteady conduction with variable thermal properties in cylindrical coordinate*

CAI Ruixian** and ZHANG Na

(Institute of Engineering Thermophysics, Chinese Academy of Sciences, Beijing 100080, China)

Received July 30, 2003; revised August 15, 2003

Abstract The analytical solutions of unsteady heat conduction with variable thermal properties (thermal conductivity, density and specific heat are functions of temperature or coordinates) are meaningful in theory. In addition, they are very useful to the computational heat conduction to check the numerical solutions and to develop numerical schemes, grid generation methods and so forth. Such solutions in rectangular coordinates have been derived by the authors. Some other solutions for 1-D and 2-D axisymmetrical heat conduction in cylindrical coordinates are given in this paper to promote the heat conduction theory and to develop the relative computational heat conduction.

Keywords: analytical solution, heat conduction, nonlinear, variable thermal property, cylindrical coordinates.

Analytical solutions of constant coefficient heat conduction equations played a key role in the early development of heat conduction[1]. Practically, various thermal properties (thermal conductivity, density and specific heat) are variable and the heat conduction process is commonly unsteady, it is not easy to derive the analytical solutions. According to the knowledge of the authors, perhaps no any algebraically explicit analytical solutions of unsteady heat conduction with variable thermal properties have been found in the open literatures so far except for two papers recently given by the authors[2,3] in rectangular coordinates. In order to develop the heat conduction theory, it is valuable to find out some new analytical solutions in cylindrical coordinates. For the same reason, the first author recently provided some algebraically explicit analytical solutions of unsteady nonlinear compressible flow[4~7] to develop aerodynamics.

Besides theoretical meaning, analytical solutions can also be applied to check the accuracy, convergence and effectiveness of various numerical computation methods and their differencing schemes, grid generation ways and so on. For example, in the fluid dynamics field, several analytical solutions which can simulate the 3-D potential flow in turbine cascades were given by the first author et al.[8], these solutions have been used successfully by scientists to check their computational methods and computer codes[8~12].

Therefore, continuing the work presented in Refs. [2, 3], some algebraically explicit analytical solutions of unsteady geometrically 1-D and 2-D axisymmetrical heat conduction equations with variable thermal properties are derived in cylindrical coordinates.

It is emphasized that the main aim of this paper is to obtain some possible explicit analytical solutions of the heat conduction equation with variable coefficients but not a specified solution for given initial and boundary conditions, therefore, the initial and boundary conditions are indeterminate before derivation and deduced from the solutions afterward. It makes the derivation procedure easier. In order to derive explicit analytical solutions, another important point is that the function of the thermal conductivity and the function of the density and the specific heat have to be matchable in some degree. Moreover, in some cases a skill (the method of separating variables with addition[6,13]) is applied to solve the equation. It is assumed that the unknown solution $\theta = \theta(t, z, r) = T(t) + Z(z) + R(r)$ replaces $\theta = (t, z, r) = T(t) \cdot Z(z) \cdot R(r)$ in the common method of separating variables. Indeed, sometimes the derivation procedure is basically a trial and error one with the help of inspiration, experience and fortune.

Actually, all solutions given in this paper can be proven easily by substituting them into the governing

* Supported by the National Natural Science Foundation of China (Grant No. 50246003) and the Major State Basic Research Development Program of China (Grant No. G20000263)

** To whom correspondence should be addressed. E-mail: crx@mail.etp.ac.cn

equation.

1 Axisymmetrical heat conduction equation with variable thermal properties in cylindrical coordinates

The unsteady symmetrical geometrically 2-D heat conduction governing equation in cylindrical coordinates is

$$\rho C_p \frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(Kr\frac{\partial \theta}{\partial r}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial \theta}{\partial z}\right), \quad (1)$$

where density ρ, specific heat C_p and thermal conductivity K are commonly inconstant. They could be functions of coordinates (r, z) or functions of temperature θ (nonlinear case), but are always positive. In addition, t is time coordinate.

For geometrically 1-D case and only considering radial variation, Eq. (1) can be simplified as

$$\rho C_p \frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(Kr\frac{\partial \theta}{\partial r}\right). \quad (1a)$$

2 Unsteady geometrically 1-D heat conduction solutions with thermal properties varying along radial direction

When $\rho C_p = f(r)$, $K = K(r)$ and applying the method of separating variables with addition $\theta = T(t) + R(r)$, substituting these relations into Eq. (1a) and separating variables, we obtain

$$T' = C_1 = \frac{\mathrm{d}}{\mathrm{d}r}[K(r)rR']/[rf(r)], \quad (2)$$

where each C_i represents different constants. From left hand side of Eq. (2), it is deduced that

$$T = C_1 t. \quad (3)$$

From right hand side of Eq. (2), it is derived that

$$R = C_1\int \frac{\int rf(r)\mathrm{d}r}{rK(r)}\mathrm{d}r. \quad (4)$$

Then the final result with thermal properties varying along radius is

$$\theta = C_1 t + C_1\int \frac{\int rf(r)\mathrm{d}r}{rK(r)}\mathrm{d}r. \quad (5)$$

Eq. (5) is a kind of analytical solution of basic heat conduction Eq. (1a) represented with integration. There will be different solutions with different thermal property distributions. However, for more evidently representing physical relation and more suitable to be a benchmark solution, it is better to have an algebraically explicit analytical solution. It can be done when $f(r)$ and $K(r)$ have a matchable relation. Some concrete cases are given in the following paragraphs.

When the common method of separating variables is applied: $\theta = T(t)\cdot R(r)$, substituting this relation as well as $\rho C_p = f(r)$ and $K = K(r)$ into Eq. (1a), we obtain

$$\frac{T'}{T} = C_{10} = \frac{1}{rf(r)R}\frac{\mathrm{d}}{\mathrm{d}r}[rK(r)R'] \quad (6)$$

and the solution is

$$\theta = C_{11}\mathrm{e}^{C_{10}t}\cdot R, \quad (7)$$

where R is the solution of the following ordinary differential equation:

$$R^* + [K(r) + rK'(r)]R'/[rK(r)] = C_{10}f(r)R/K(r). \quad (8)$$

Since the solution (7) obtained by the common method of separating variables includes an ordinary differential equation (8), but the solution (5) obtained by the method of separating variables with addition only includes integrations, the last method can derive algebraically explicit analytical solutions easier.

When thermal properties are only functions of radius, Eq. (1a) is linear, then the sum of solutions of both separating methods is the solution of Eq. (1a) also.

2.1 Solutions with $\rho C_p = f(r)$ and $K(r) = k/r$ (k is a constant)

According to Eqs. (5), (7) and (8), and replacing C_{10} and C_{11} with C_4 and C_5, the solutions can be simplified as

$$\theta = C_1 t + C_1\iint rf(r)\mathrm{d}r\mathrm{d}r/k + C_5\mathrm{e}^{C_4 t}\cdot R(r), \quad (9)$$

where R is the solution of the following ordinary differential equation:

$$R''(r) = C_4 rf(r)R(r)/k. \quad (9a)$$

Actually there are infinite $f(r)$ which can derive algebraically explicit analytical solutions from Eqs. (9) and (9a). Some examples are illustrated as follows:

2.1.1 Solution with $\rho C_p = f(r) = m/r$ (m is a constant)

Substituting $\rho C_p = m/r$ into Eqs. (9) and (9a), it can be derived that

$$\theta = C_1 t + C_1 m(r^2/2 + C_2 r + C_3)/k + C_5\cdot\exp(C_4 t + \sqrt{C_4 m/kr})$$

$$+ C_6\exp(C_4 t - \sqrt{C_4 m/kr}) \quad \text{(when } C_4 > 0), \tag{10a}$$

$$\theta = C_1 t + C_1 m(r^2/2 + C_2 r + C_3)/k + \exp(C_4 t)[C_5\sin(\sqrt{-C_4 m/kr}) + C_6\cos(\sqrt{-C_4 m/kr})] \quad \text{(when } C_4 < 0), \tag{10b}$$

where m and p have to be positive owing to physical conditions, but C_4 can be positive or negative. As an example, the initial and boundary conditions of this solution when $C_5 = 0 = C_6$ (equivalent to the solution derived with the method of separating variables with addition only) can be given with Eq. (10) as follows: $t = 0$, $\theta = C_1 m(r^2/2 + C_2 r + C_3)/k$, it means that the initial temperature distribution is uneven; $r = r_1$, $\theta = C_1 t + C_1 m(r_1^2/2 + C_2 r_1 + C_3)/k$ and $r = r_2$, $\theta = C_1 t + C_1 m(r_2^2/2 + C_2 r_2 + C_3/k)$, it means the boundary conditions are unsteady. The initial and boundary conditions of other solutions given in the following paragraphs can be given similarly, each solution corresponds to its own conditions. Since the expressions of thermal properties in this solution are a fraction with denominator r, it can only be used for a circular ring region. Most solutions in the following paragraphs have the same restriction.

Owing to the limitation of space, the typical curves of various solutions in this paper are not given.

2.1.2 Solution with $\rho C_p = f(r) = 1/[r(\pm C_4 r^2/2 + C_2 r + C_3)]$

Similar to the previous paragraph, the solution can be derived as follows:

$$\theta = C_1 t + \frac{C_1}{C_4 kp}\left\{[g(r) \mp p]\ln\left[\frac{g(r) \mp p}{e}\right] - [g(r) \pm p]\ln\left[\frac{\pm g(r) + p}{e}\right]\right\} + C_6 r + C_7 + C_5\exp(\pm C_4 kt)\left[\pm\frac{C_4}{2}r^2 + C_2 r + C_3\right], \tag{11}$$

where $g(r) = C_4 r - C_2$ and $p = \sqrt{C_2^2 + 2C_3C_4}$; in addition, $g(r)$ has to be larger or smaller than p when the $\pm$ mark in front of C_4 in $f(r)$ is chosen positive or negative, otherwise there will be logarithm of a negative. C_4 cannot be zero, C_2 and C_3 cannot be zero simultaneously, otherwise the denominator in Eq. (11) would be zero.

When $C_4 = 0$, $\rho C_p = 1/[r(C_2 r + C_3)]$, the solution can be deduced in a similar way:

$$\theta = C_1 t + C_1[(C_2 r + C_3)\ln(C_2 r + C_3) - (C_2 r + C_3)]/(kC_2^2) + C_5(C_2 r + C_3). \tag{11a}$$

The solution with $C_2 = 0 = C_3$ is given in the next paragraph.

2.1.3 Solution with $\rho C_p = f(r) = 1/[mr(r + C_6)^2]$

The solution can be obtained with a not very difficult trial and error method as follows:

$$\theta = C_1 t - C_1\ln(r + C_6)/(km) + C_2 r + C_3 + C_5 e^{C_4 t}(r + C_6)^{[1 \pm \sqrt{1 + 4C_4/(km)}]/2}. \tag{12}$$

The variation of ρC_p along r is too rapid, such condition is very rare in practice. However, it can still be a benchmark solution of computational heat conduction. It is evident that the constant m has to be positive otherwise ρC_p is negative; in addition, C_4 has to be larger than $-km/4$, otherwise there would be an imaginary number in the index.

2.1.4 Solution with $\rho C_p = f(r) = C_2\sec^2[\sqrt{C_2C_4/(2k)}(r + C_3)]/r$

The solution is

$$\theta = C_1 t - \frac{2C_1}{C_4}\ln\left\{\cos\left[\sqrt{\frac{C_2C_4}{2k}}(r + C_3)\right]\right\} + C_6 r + C_7 + C_5 e^{C_4 t}\cdot\sqrt{\frac{2C_2 k}{C_4}}\tan\left[\sqrt{\frac{C_2C_4}{2k}}(r + C_3)\right]. \tag{13}$$

By selecting the value of various constants, the variation of ρC_p can be increasing or decreasing along r direction. C_2 and C_4 of this solution have to be larger than zero in order to avoid unreasonable ρC_p or zero denominator.

2.1.5 Solution with $\rho C_p = f(r) = C_3 r^m$

Commonly the variation of ρC_p in radial direction is not serious, then the absolute value of m is small.

Using the method of separating variables with addition, a solution can be derived with abovementioned ρC_p distribution.

$$\theta = C_1 t + C_1 C_3 r^{m+3}/[k(m+2)(m+3)] + C_2 r + C_8. \tag{14}$$

Actually, when $m = -1$, the solution is the degeneration form of Eq. (10a) or (10b). When $m = -2$ or -3, Eq. (14) is ineffective, and the special solution of such cases are

$$\theta = C_1 t + (C_1 C_3 r \ln r)/k + C_2 r + C_8 \quad (14a)$$

and

$$\theta = C_1 t - (C_1 C_3 \ln r)/k + C_2 r + C_8 + C_5 e^{C_4 t} r^{1 \pm \sqrt{1-4C_3C_4/k}/2} \quad (14b)$$

respectively. Eq. (14b) is a special case of Eq. (12) also.

When $m = 0$, it means $\rho C_p = \text{Const}$ and $K \sim 1/r$, the solution of this case is

$$\theta = C_1 t + C_1 C_3 r^3/(6k) + C_2 r + C_8. \quad (14c)$$

From Eq. (14c), it is understood that even ρC_p is a constant there is still a solution with very high temperature variation in radial direction; of course it needs some defined boundary conditions and $K \sim 1/r$.

2.1.6 Solution with $\rho C_p = f(r) = C_6 e^{C_3 r}$

Similar to the previous paragraph, commonly the variation of ρC_p in radial direction is not serious, so the absolute value of C_3 is small.

Using the method of separating variables with addition, the solution is deduced as

$$\theta = C_1 t + C_2 + C_4 r + C_1 C_6 e^{C_3 r}(r - 2/C_3)/(k C_3^2), \quad (15)$$

where $C_3 \neq 0$ (equivalent to $\rho C_p \neq \text{Const}$). Otherwise there would be a zero denominator.

When $C_3 = 0$, the solution is Eq. (14c).

2.2 Solution with $\rho C_p = f(r)$ and $K = g(r)/r$

Similar to the derivation of Eqs. (9) and (9a), the solution is

$$\theta = C_1 t + C_1 \int \frac{\int r f(r) \mathrm{d}r}{g(r)} \mathrm{d}r + C_{11} e^{C_{10} t} R(r), \quad (16)$$

where $R(r)$ is the solution of the following ordinary differential equation

$$R'' + g'(r) R'/g(r) - C_{10} r f(r) R/g(r) = 0. \quad (17)$$

Of course, Eqs. (9) and (9a) are the special cases of Eqs. (16) and (17). There are many combinations of $f(r)$ and $g(r)$ which can obtain algebraically explicit analytical solutions of Eqs. (16) and (17). Some examples are given as follows.

2.2.1 Solution with $\rho C_p = f(r) = C_6 e^{C_5 r}$ and $K = C_4 e^{C_3 r}/r$

Since the variation of ρC_p and K in radial direction is commonly small, the absolute values of C_5 and C_3 are small.

Substituting the expressions of ρC_p and K into Eq. (16) and applying only the method of separating variables with addition ($C_{11} = 0$), we have

$$\theta = C_1 t + C_2 + \frac{C_1 C_6}{C_4 C_5} \frac{e^{(C_5 - C_3) r}}{C_5 - C_3} \left(r - \frac{1}{C_5 - C_3} - \frac{1}{C_5} \right) - \frac{C_7}{C_3} e^{-C_3 r}, \quad (18)$$

where C_4 and C_6 are positive values to satisfy ρC_p and K being positive; C_3 and C_5 cannot be zero and $C_3 \neq C_5$, otherwise there would be a zero denominator; C_2 has to be large enough to maintain the temperature $\theta > 0$ in the interested region.

When $C_3 = 0$, $C_5 = 0$ or $C_3 = C_5 \neq 0$, the special solutions are derived as follows.

For $C_3 = 0$, (i.e. $K \sim 1/r$), the solution is actually the same as that in paragraph 2.1.6, the k and C_3 in this paragraph is equivalent to C_4 and C_5 in paragraph 2.1.6.

For $C_5 = 0$, (i.e. $\rho C_p = \text{Const} = C_6$) and $C_3 \neq 0$, we obtain

$$\theta = C_1 t/C_6 + C_2 - C_1 e^{-C_3 r}[r^2 + 2r/C_3 + 2/C_3^2]/(2 C_3 C_4) - C_7 e^{-C_3 r}/C_3. \quad (18a)$$

For $C_3 = C_5 \neq 0$, using both methods of separating variables, it is obtained from Eqs. (16) and (17) that

$$\theta = C_1 t + C_2 + C_1 C_6 r/(C_3 C_4) - C_7 e^{-C_3 r}/C_3 + \theta_1, \quad (18b)$$

where the θ_1 expressions are

$$\theta_1 = e^{C_{10} t}[C_8 e^{(\sqrt{C_3^2 + 4C_6C_{10}/C_4} - C_3) r/2} + C_9 e^{-(\sqrt{C_3^2 + 4C_6C_{10}/C_4} + C_3) r/2}]$$

when $C_3^2 + 4 C_6 C_{10}/C_4 > 0$, (18c)

$$\theta_1 = e^{C_{10} t - C_3 r/2} \{ C_8 \cos[\sqrt{-(C_3^2 + 4C_6 C_{10}/C_4)}\, r/2] + C_9 \sin[\sqrt{-(C_3^2 + 4C_6C_{10}/C_4)}\, r/2] \}$$

when $C_3^2 + 4 C_6 C_{10}/C_4 < 0$,

$$\theta_1 = e^{C_{10} t - C_3 r/2}(C_8 + C_9 r)$$

when $C_3^2 + 4C_6C_{10}/C_4 = 0$.

2.2.2 Solutions with $\rho C_p = f(r) = C_3 r^m$ and $K = C_4 r^{l-1}$

As mentioned above, the variation of ρC_p and K in radial direction is small, then the absolute value of m is small and $l \approx 1$, C_3 and C_4 are positive.

Substituting the expressions of ρC_p and K into Eq. (16) and using the method of separating variables with addition ($C_{11} = 0$), it is deduced that

$$\theta = C_1 t + C_2 + C_1 C_3 r^{m+3-l} / \lfloor C_4(m+2)(m+3-l) - C_7 r^{1-l}/(1-l) \rfloor, \tag{19}$$

where $m \neq = -2$, $l \neq m+3$ and $l \neq 1$. Otherwise there would be a zero denominator.

When $m = -2$, with a similar derivation, a special solution can be obtained as the following:

$$\theta = C_1 t / C_3 + C_2 + C_1 r^{1-l}[\ln r - 1/(1-l)]/[C_4(1-l)] + C_7 r^{1-l}/(1-l). \tag{19a}$$

Eq. (19a) is still ineffective when $K = \text{Const}$ ($l = 1$). In this case, another solution can be deduced directly from Eqs. (16) and (17) using both methods of separating variables as follows:

$$\theta = C_1 t / C_3 + C_2 + C_1 (\ln r)^2/(2C_4) + C_7 \ln r + C_{11} e^{C_{10} t / C_3} r^{\pm \sqrt{C_{10}/C_4}}. \tag{19b}$$

The special solution with only $l = 1$ ($m \neq \pm 2$) is

$$\theta = C_1 t / C_3 + C_2 + C_1 r^{m+2}/[C_4(m+2)^2] + C_5 \ln r. \tag{19c}$$

When $l = m + 3$, the following explicit analytical solution is derived with a similar procedure:

$$\theta = C_1 t + C_2 + C_1 C_3 \ln r/[C_4(l-1)] + C_7 r^{1-l}/(1-l) + C_{11} e^{C_{10} t} r^{[1-l \pm \sqrt{(l-1)^2 + 4C_{10}C_3/C_4}]/2}. \tag{19d}$$

2.2.3 Solution with $\rho C_p = p e^{lr}/r$ and $K = k e^{lr}/r$

Applying both methods of separating variables, a solution is derived as

$$\theta = C_1 t + C_1 pr/kl + C_2 + \theta_1, \tag{20}$$

where the expression of θ_1 is

$$\theta_1 = e^{C_{10} t}[C_6 e^{(-l + \sqrt{l^2 + 4C_{10}p/k})r/2} + C_7 e^{-(l + \sqrt{l^2 + 4C_{10}p/k})r/2}]$$

when $l^2 > -4C_{10}p/k$,

$$\theta_1 = e^{C_{10} t - lr/2}[C_6 \cos(\sqrt{-4C_{10}p/k - l^2}\, r/2) + C_7 \sin(\sqrt{-4C_{10}p/k - l^2}\, r/2)] \tag{20a}$$

when $l^2 < -4C_{10}p/k$,

$$\theta_1 = e^{C_{10} t - lr/2}(C_6 + C_7 r)$$

when $l^2 = -4C_{10}p/k$.

If applying the method of separating variables with addition only ($C_6 = 0 = C_7$), a very simple temperature distribution and its initial and boundary conditions are obtained—linear relations with both time and geometric coordinates, it is a special characteristic of this solution.

2.3 Deriving analytical solutions in cylindrical coordinates with known analytical solutions in rectangular coordinates

Comparing the governing equation (1a) in cylindrical coordinates with the following governing equation (21) in rectangular coordinates,

$$\rho C_p \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial \theta}{\partial x} \right). \tag{21}$$

It can be concluded that for the analytical solutions $\theta(t, x)$ in rectangular coordinates satisfying Eq. (21) obtained in Ref. [3], if we replace x, $\rho C_p(x)$ and $K(x)$ in these solutions in rectangular coordinates with r, $r\rho C_p(r)$ and $rK(r)$, then some analytical solutions in cylindrical coordinates satisfying governing equation (1a) can be obtained.

For example, for the 4th solution in Ref. [3], choosing the constants $l = n = 0$, then an algebraically explicit analytical solution in cylindrical coordinates with volumetric specific heat $\rho C_p = f(r) = pr + m$ and thermal conductivity $K = jr + k$ can be obtained directly

$$\theta = C_1 t + C_1[pr^2 + (3jm - 2kp)r/j + (2k^2 p - 3jkm)\ln(jr^2 + kr)/(2j^2)]/(6j) + C_2 \ln |jr/(jr + k)|. \tag{22}$$

This solution can be used for solid cylinders as well. It can be derived with the approach given in Section 2.2 also.

3 Nonlinear unsteady geometrically 1-D heat conduction solutions

When volumetric specific heat ρC_p and /or thermal conductivity K are functions of temperature, the governing equation (1a) becomes nonlinear. It is more difficult to derive algebraically explicit analytical solutions. In addition, the solutions in this case cannot be superposed owing to the nonlinearity.

Some algebraically explicit analytical solutions for nonlinear cases are given as follows.

3.1 Solutions when volumetric specific heat is a function of temperature

The derivation approach is about the same as previously described, however, the matching of thermal properties has to be paid more attention to.

3.1.1 Solution with volumetric specific heat being a power function of temperature and thermal conductivity being a constant

The given conditions are $\rho C_p = p\theta^m$ and $K = k =$ Const. Substituting these conditions into governing equation (1a) and applying the common method of separating variables $\theta = T(t)\cdot R(r)$, we have

$$\frac{p}{k}T^{m-1}T' = C_1 = \frac{R''}{R^{m+1}} + \frac{R'}{rR^{m+1}}. \qquad (23)$$

From left hand side of Eq. (23), it is easy to deduce

$$T = \left[\frac{C_1 km}{p}(t + C_2)\right]^{1/m}. \qquad (24)$$

The right hand side of Eq. (23) is a second order ordinary differential equation including independent variable r, dependent variable R and its first order and second order derivatives, and in mathematic handbooks there is not an already known general method to solve such equation. However, according to the expressions of volumetric specific heat and Eq. (23), it can be guessed that the R is a power function of r. Assuming $R = \alpha r^{\beta}$ (α and β are constants to be determined) and substituting R expression into Eq. (23), the final result is

$$R = \left(\frac{4}{C_1 m^2}\right)^{1/m} \cdot r^{-2/m}. \qquad (25)$$

Multiplying Eqs. (24) and (25), an algebraically explicit analytical solution of nonlinear equation is obtained

$$\theta = \left[\frac{4k(t + C_2)}{mpr^2}\right]^{1/m}. \qquad (26)$$

When m is larger, the variation of specific heat with temperature is more serious. Then the temperature variation with time and geometry coordinates is smaller. Solution (26) represents this physical phenomenon.

Since the specific heat and thermal conductivity cannot be zero, and the volumetric specific heat is variable, the constants k, p and m cannot be zero.

3.1.2 Solution with volumetric specific heat being power function of temperature and thermal conductivity being function of radius

If the given conditions are $\rho C_p = p\theta^m$ and $K = kr$, the material varies in radial direction which introduces variable thermal conductivity. Applying the approach similar to the one mentioned above, it is derived that

$$\theta = \left[\frac{(1-m)k(t + C_2)}{mpr}\right]^{1/m}. \qquad (27)$$

If the thermal conductivity is inversely proportional to r: $K = k/r$, the solution can be deduced as

$$\theta = \left[\frac{3k(3+m)(t + C_2)}{mpr^3}\right]^{1/m}. \qquad (28)$$

Comparing three algebraically explicit analytical solutions with different variation of thermal conductivity with radius Eqs. (26) ~ (28), some relations can be found, for example, the influence of variation of index of r on the solutions. It can be estimated that when the variation of K with r is between the linear and inverse proportion and not a constant, the solution condition will be a case between the abovementioned three solutions. However, it has to be mentioned that their initial and boundary conditions are different.

3.2 Deriving analytical solutions in cylindrical coordinates with known analytical solutions in rectangular coordinates

Similar to what described in Section 2.3, for nonlinear heat conduction equations it is also able to use known analytical solutions in rectangular coordinates to obtain analytical solutions in cylindrical coordinates. In this case, both volumetrically specific heat and thermal conductivity are functions of temperature and radius. Owing to the limitation of space, only one example is given as follows:

$$\rho C_p = m e^{l\theta}/r,$$
$$K = k e^{l\theta}/r,$$
$$\theta = C_1 kt/m \pm \sqrt{C_1/lr} + C_2, \qquad (29)$$

or

$$\theta = C_1 kt/m + \ln\{\cosh[\sqrt{C_1 l}(r + C_3)]\}/l + C_2.$$

Two different solutions given by Eq. (29) are corresponding to different initial and boundary conditions. In addition, C_1 has to be larger than zero. In the first solution of Eq. (29) θ is a linear function of t and r, it is very suitable to be a benchmark solution of computational heat transfer.

4 Analytical solution of unsteady geometrically 2-D heat conduction

The governing equation (1) of unsteady geometrically 2D heat conduction is actually a mathematical 3-D problem. Therefore, it is more difficult to derive its algebraically explicit analytical solutions. Some solutions are given in following paragraphs for variable thermal properties (both volumetric specific heat and thermal conductivity are functions of coordinates) and nonlinear case (thermal properties are functions of temperature also).

4.1 Analytical solutions for variable thermal properties

4.1.1 Solution with thermal conductivity being constant

If $K = k = \text{Const}$, and $\rho C_p = R(r) + Z(z)$, i.e. the volumetric specific heat varies in both radial and axial directions owing to the variation of the material composition with neglecting the variation of thermal conductivity, then a kind of solutions can be derived with the method of separating variables with addition as follows:

$$\theta = \frac{C_1}{k}\left[\int \frac{1}{r}\int rR\,\mathrm{d}r\,\mathrm{d}r + \iint Z\,\mathrm{d}z\,\mathrm{d}z\right] + C_1 t + \frac{C_2}{2}\left(\frac{r^2}{2} + C_6\ln r - z^2\right) + C_4 z + C_5. \tag{30}$$

Since $R(r)$ and $Z(z)$ are arbitrary functions, the number of solutions of Eq. (30) is infinite. As an example, if the volumetric specific heat variations in both radial and axial directions are linear, $R = l + mr$ and $Z = n + pz$, then an algebraically explicit analytical solution can be expressed as:

$$\theta = \frac{C_1}{k}\left(\frac{lr^2}{4} + \frac{mr^3}{9} + \frac{n}{2}z^2 + \frac{p}{6}z^3\right) + C_1 t + \frac{C_2}{2}\left(\frac{r^2}{2} + C_6\ln r - z^2\right) + C_4 z + C_5. \tag{30a}$$

When degenerating into geometrical 1-D case, $Z(z)$ in Eq. (30) has to be equal to zero. Eq. (19c) in paragraph 2.2.2 can be deduced from Eq. (30a) and it is a special case.

4.1.2 Solution with pure linear temperature distribution

If there is the following relation between the volumetric specific heat and the thermal conductivity

$$\rho C_p = \left[C_2\left(\frac{\partial K}{\partial r} + \frac{K}{r}\right) + C_3\frac{\partial K}{\partial z}\right]\Big/ C_1, \tag{31}$$

then an extremely simple solution with temperature being a linear function of time as well as geometrical coordinates can be derived as follows:

$$\theta = C_1 t + C_2 r + C_3 z + C_4. \tag{32}$$

Since the function expressions of K (and then ρC_p) can be infinite and there is only derivation in Eq. (31), therefore, the deduction of Eqs. (31) and (32) means it is very easy to find out the matching of ρC_p and K which can obtain a pure linear temperature distribution along with both time and geometrical coordinates. The solution in paragraph 2.2.3 (excluding θ_1) is actually a special case of this paragraph.

4.2 Analytical solutions of nonlinear case

Similar to that described in Section 3.2, it is able to derive nonlinear unsteady geometrically 2-D heat conduction solutions in cylindrical coordinates from known solutions in rectangular coordinates in Ref. [2]. Some examples are given here.

4.2.1 Solution with $\rho C_p = m\mathrm{e}^{l\theta}/r$ and $K = k\mathrm{e}^{l\theta}/r$

In this case, the variations of volumetric specific heat and thermal conductivity with temperature have the same rule, the variations of both thermal properties with radial coordinate have the same rule also. The absolute value of l is small since the variations of thermal properties are commonly small. An algebraically explicit analytical solution can be found as the following:

$$\theta = C_1 kt/m \pm \sqrt{C_3/l}\,r \pm \sqrt{(C_1 - C_3)l}\,z + C_2. \tag{33}$$

In this solution, temperature variation is linear along with all three independent variables, it is a good benchmark solution. By the way, the first solution of Eq. (29) is a special case of this solution.

Another possible solution is

$$\theta = C_1 kt/m + \ln|\cosh[\sqrt{C_3 l}(r + C_2)]|/l$$

$$+ \ln\{\cosh[\pm \sqrt{(C_1 - C_3)l}(z + C_4)]\}/l + C_5. \quad (34)$$

The second solution of Eq. (29) is the 1-D simplified case of Eq. (34).

4.2.2 Solution with $\rho C_p = m\theta^l / r$ and $K = k\theta^l / r$

Similar to previously mentioned, the variation rules of volumetric specific heat and thermal conductivity with temperature and radius are the same. A possible solution is as follows:

$$\theta = \exp[C_1 k(t + C_2)/m + \sqrt{C_3/(l+1)}(r + C_4) + \sqrt{(C_1 - C_3)/(l+1)}(z + C_5) + C_6]. \quad (35)$$

Eq. (35) is not suitable when $l = -1$. For $l = -1$, the expression has to be changed into

$$\theta = \exp\lfloor C_1 k(t + C_2)/m + C_1 C_3 (r + C_4)^2/2 + C_1(1 - C_3)(z + C_5)^2/2 \rfloor. \quad (36)$$

5 Conclusion

About 20 kinds of algebraically explicit analytical solutions for unsteady heat conduction equations in axisymmetrical cylindrical coordinates with variable thermal properties are given. They can be the benchmark solutions to develop computational heat transfer and check the numerical solutions.

Acknowledgement The authors are grateful to Ms. Wu Jie for her help in preparing this manuscript.

References

1 Ozisik, M N. Heat Conduction. Scranton: International Textbook Co., 1983.

2 Cai, R. et al. Some algebraically explicit analytical solution of unsteady nonlinear heat conduction. ASME J Heat Transfer, 2001, 123(6): 1189.

3 Cai, R. et al. Explicit analytical solutions of unsteady conduction with variable thermodynamic properties. J. Engg. Thermophysics (in Chinese), 2002, 23(3): 327.

4 Cai, R. Some analytical solutions of unsteady compressible flow. ASME J. Fluids Engg., 1998, 120(4): 760.

5 Cai, R. et al. Analytical solutions of inviscid compressible perfect gas flow. J. Engg. Thermophysics (in Chinese), 1998, 19(4): 439.

6 Cai, R. Derivation of explicit analytical solutions for nonlinear unsteady compressible isentropic flow. J. Engg. Thermophysics (in Chinese), 2001, 22(2): 159.

7 Cai, R. et al. Explicit analytical wave solutions of unsteady 1-D ideal gas flow with friction and heat transfer. Science in China (Ser. A), 2001, 44(4): 414.

8 Cai, R. et al. Some analytical solutions applicable to verify 3-D numerical methods in turbomachines. IMechE Conf. Publications C 80/84, 1984: 255.

9 Zhu, S. et al. An investigation of numerical method of 3-D flow field in the blade passage of centrifugal pump (compared with analytical solution). J. Engg. Thermophysics (in Chinese) 1988, 9 (3): 242.

10 Xu, J. et al. 3-D incompressible flow solution of an axial compressor using pseudostream-functions formulation. 1989, ASME paper 89-GT-319.

11 Gong, Y. et al. 3-D MSLM-a new engineering approach to the inverse problem of 3D cascade. 1989, ASME paper 89-GT-48.

12 Shen, M. et al. Calculation of 3-D transonic flow of turbomachinery with the generalized von Mises coordinates system. Science in China (Ser. A), 1996, 39: 1084.

13 Cai, R. et al. Unsteady 1-D analytical solutions for bioheat transfer equations. Progress in Natural Science, 1998, 8(6): 733.